Combination of Electroweak and QCD Radiative Corrections to Neutral-Current Drell-Yan Processes at Hadron Colliders DRAFT - only for discussion within CTEQ4LHC WG

I. EW AND QCD CORRECTIONS TO Z PRODUCTION

- Proposal 1:

  Inclusion of weak corrections using the effective (sometimes also called improved) Born approximation (EBA) (see Section III B for details). QED radiation included with PHOTOS (leading multiple photon radiation) or ZGRAD2 (complete 1-loop).

  Use of $\alpha(G_\mu)$ input scheme, effective vector and axial vector couplings from ZGRAD2 or use measured values for $\sin^2\theta_{eff}$ and Z width.

- Proposal 2:

  MC@NLO and PHOTOS

  or

  MC@NLO and HORACE (see also Les Houches report)

  or

  ResBos and QED corrections (see also Les Houches report)

  See Ref. [1] for a discussion of the numerical differences.

  The consensus seems to be that for 1% (and better) precision one needs to do better.

- Proposal 3:

  Implementation of complete EW 1-loop corrections in FEWZ or of NLO (and maybe also NNLO) QCD corrections in ZGRAD2.

This will be discussed in the context of the CTEQ4LHC workshop. Moreover: Inclusion of 2-loop EW Sudakov logs and multiple final-state photon radiation.

II. ELECTROWEAK INPUT SCHEMES

In electroweak (EW) calculations one has the choice between different EW input schemes. The input scheme determines which EW parameters are input parameters and which ones are calculated. The different options may agree at the strict one-loop level, but differ in the treatment of higher-order (ie beyond 1-loop) contributions. Thus, in EW 1-loop calculations usually the option is used that provides an improvement by including some of the two-loop (and higher order) terms.

The different EW input schemes that are discussed by most EW authors are (see also discussion in the TeV4LHC report and in Ref. [2]):

- $\alpha(0)$ scheme:

  The fine structure constant $\alpha(0)$ is used in all parts of the calculation, i.e. in the Born cross sections and EW 1-loop corrections. EW input parameters are:

  $$\alpha(0), M_W, M_Z, M_H, m_f$$

  This scheme is mostly used in comparisons of different EW calculations to simplify the setup. It is not recommended for use in EW 1-loop calculations, since the cross sections will depend on light quark masses.

- $\alpha(M_Z)$ scheme:

  $\alpha(M_Z)$ is used only in the calculation of the Born matrix element when used in EW 1-loop calculations. EW 1-loop corrections are still evaluated at $\mathcal{O}(\alpha(0))$ so that the correct energy scale for photon radiation is used. One could use $\alpha(M_Z)$ also in the 1-loop part, then some factorized parts of two-loop terms and higher are introduced. Set of EW input parameters:

$$\alpha(M_Z), M_W, M_Z, M_H, m_f$$

Here the running of QED α is taken into account, ie the photon selfenergy contribution. The advantage is that the light quark-mass dependence cancels in EW 1-loop calculations.

- $\alpha(G_\mu)$ scheme (prefered scheme in EW 1-loop calculations):

  $\alpha(G_\mu)$ is used only in the calculation of the Born matrix element EW 1-loop corrections are still evaluated at $\mathcal{O}(\alpha(0))$ so that the correct energy scale for photon radiation is used. In leading-order calculations $\alpha(G_\mu)$ is calculated as follows:

$$\alpha(G_\mu) = \frac{\sqrt{2}}{\pi} G_\mu M_W^2 \sin^2\theta_w$$

  and in EW 1-loop calculations one uses:

$$\alpha(G_\mu) = \frac{\sqrt{2}}{\pi} G_\mu M_W^2 \sin^2\theta_w (1-\Delta r)$$

  Here the corrections to the muon decay are included as described by Δr.Δr is known beyond one-loop. Set of EW input parameters:

$$\alpha(G_\mu), M_W, M_Z, m_f$$

  and M_H is calculated from $\Delta r(M_W, M_H, ...)$. Alternatively one can use:

$$\alpha(G_\mu), M_H, M_Z, m_f$$

  and M_W is calculated from $\Delta r(M_W, M_H, ...)$. Again the running of QED α is taken into account as part of the Δr contribution, and the light quark-mass dependence cancels in EW 1-loop calculations. This is the prefered scheme for EW 1-loop calculations and is the default scheme in ZGRAD2 (see also description in Section III C).

Note that in all schemes described above one uses $c_w = M_W/M_Z$.

Another important issue is the treatment of the Z resonance. One can either use an s-dependent or constant width approach (see discussion below). Also, the width should be calculated at the same order as the matrix element. Another option is to use the measured Z width.

III. ELECTROWEAK RADIATIVE CORRECTIONS TO NEUTRAL CURRENT DRELL-YAN PROCESSES

Taken from Ref. [3]. The parton-level differential Born cross section to charged lepton pair production via photon and Z boson exchange in quark-antiquark annihilation ($l = e, \mu$)

$$q(p) + \bar{q}(\bar{p}) \to \gamma, Z \to l^+(k_+) + l^-(k_-)$$

can be written as follows

$$\mathrm{d}\hat{\sigma}^{(0)} = \mathrm{dP_{2f}}\, \frac{1}{12} \sum |A^0_\gamma + A^0_Z|^2(\hat{s}, t, u)\ , \tag{1}$$

where $\sum$ denotes the summation over the spin and color degrees of freedom of the initial and final state fermions and $\mathrm{dP_{2f}}$ is the two-particle phase space element. The factor $1/12$ results from averaging over the quark degrees of freedom. The matrix elements A^0_γ and A^0_Z describe the photon and Z boson exchange processes, respectively, at lowest order in perturbation theory. In terms of the kinematical variables of the parton system

$$\hat{s} = (p + \bar{p})^2, \quad t = (p - k_+)^2, \quad u = (p - k_-)^2 \tag{2}$$

the Born matrix elements squared for massless external fermions read

$$\begin{aligned}
\sum |\mathcal{A}^0_\gamma|^2 &= 8\, Q_q^2\, Q_l^2\, (4\pi\alpha)^2\, \frac{(t^2 + u^2)}{\hat{s}^2} \\
\sum |\mathcal{A}^0_Z|^2 &= 8\, \frac{|\chi(\hat{s})|^2}{\hat{s}^2}\, [(v_q^2 + a_q^2)(v_l^2 + a_l^2)(t^2 + u^2) - 4 v_q a_q v_l a_l\, (t^2 - u^2)] \\
\sum 2\mathcal{R}e(\mathcal{A}^0_Z \mathcal{A}^{0*}_\gamma) &= 16\, Q_q\, Q_l\, a_q\, a_l\, (4\pi\alpha)\, [v_q v_l (t^2 + u^2) - a_q a_l (t^2 - u^2)]\, \frac{\mathcal{R}e\chi(\hat{s})}{\hat{s}^2}
\end{aligned} \tag{3}$$

with v_f and a_f parametrizing the $Zf\bar{f}$ ($f = l, q$) couplings

$$v_f = \frac{1}{2 s_w c_w}(I_f^3 - 2 s_w^2 Q_f), \quad a_f = \frac{I_f^3}{2 s_w c_w} \; . \tag{4}$$

Q_f and I_f^3 denote the charge and third component of the isospin quantum numbers of the fermion, respectively, and $s_w \equiv \sin\theta_w, c_w \equiv \cos\theta_w$ with θ_w being the electroweak mixing angle. $\alpha \equiv \alpha(0)$ is the electromagnetic fine structure constant. Owing to the instability of the Z boson, the pole in the Z boson propagator is regularized by assuming a complex Z boson mass M_c

$$\chi(\hat{s}) = 4\pi\alpha \, \frac{\hat{s}}{(\hat{s} - M_c^2)} \; . \tag{5}$$

In a perturbative calculation of the Z propagator, a Dyson resummation of one-particle-irreduzible (1PI) (renormalized) Z self energies is performed. The imaginary part of M_c^2 is related to the Z decay width Γ_Z by unitarity and $\mathcal{R}e M_c^2(\hat{s} = M_Z^2) = M_Z^2$ with the physical Z boson mass M_Z, which in the ON-SHELL renormalization scheme [16] is equal to the renormalized mass. The Dyson resummation introduces the well-known problem of finding a definition of the mass and decay width of the Z boson and a gauge invariant description of the scattering amplitude order-by-order in perturbation theory. Mainly two approaches have been discussed in the literature, the constant-width [7] and s-dependent width [8,9] approach. As discussed in detail in [9] for a description of the Z resonance at $\mathcal{O}(\alpha)$ accuracy the $\mathcal{O}(\alpha^2)$ contributions to the imaginary part of M_c^2 must be taken into account, so that a consistent expansion in $\mathcal{O}(\alpha^2)$ yields

$$\mathcal{I}m M_c^2(\hat{s}) = \mathcal{I}m\Big(\hat{\Sigma}^Z(\hat{s})[1 + \mathcal{R}e\hat{\Pi}^Z(M_Z^2)] + \hat{\Sigma}^Z_{(2)}(\hat{s}) - \frac{(\hat{\Sigma}^{\gamma Z}(\hat{s}))^2}{\hat{s} + \hat{\Sigma}^{\gamma}(\hat{s})}\Big) \tag{6}$$

with $\hat{\Pi}^Z$ of Eq. 15 and $\hat{\Sigma}^{Z,\gamma,\gamma Z}$ of Eqs. B2,B1,B3 denoting the (renormalized) self energy insertions into the Z and photon propagators. The last term in Eq. 6 takes into account that the photon and Z boson do not propagate independently beyond leading order in perturbation theory. The transverse parts of the renormalized one and two-loop corrected 1PI Z self energies are denoted by $\hat{\Sigma}^Z$ and $\hat{\Sigma}^Z_{(2)}$, respectively. The evaluation of $\mathcal{I}m M_c^2$ at $\hat{s} = M_Z^2$ corresponds to a Laurent expansion around the complex pole and leads to a description of the Z resonance with a constant Z boson width $M_c^2 = M_Z^2 - iM_Z\Gamma_Z^{(0+1)}$. The consideration of the s-dependence of the imaginary parts of the 1PI self energies by $\mathcal{I}m\hat{\Sigma}(\hat{s}) = \hat{s}/M_Z^2 \mathcal{I}m\hat{\Sigma}(M_Z^2)$ leads to the description with an s-dependent width $M_c^2 = M_Z^2 - i\hat{s}/M_Z\Gamma_Z^{(0+1)}$. Both descriptions are related by a transformation of the parameters of the Z resonance, M_Z, Γ_Z and the residue of the complex pole, and thus are equivalent [10]. We choose the s-dependent width approach. The one-loop corrected Z boson decay width, $\Gamma_Z^{(0+1)}$, is discussed in Appendix A.

The electroweak $\mathcal{O}(\alpha)$ corrections to neutral-current Drell-Yan processes naturally decompose into QED and weak contributions, i.e. they build gauge invariant subsets, and thus can be discussed separately. The observable NLO cross section is obtained by convoluting the parton cross section with the quark distribution functions $q(x, Q^2)$ ($\hat{s} = x_1 x_2 S$)

$$\mathrm{d}\sigma(S) = \int_0^1 dx_1 dx_2\, q(x_1, Q^2)\, \bar{q}(x_2, Q^2) [\mathrm{d}\hat{\sigma}^{(0+1)}(\hat{s}, t, u) + \mathrm{d}\hat{\sigma}_{\mathrm{QED}}(\mu_{\mathrm{QED}}^2, \hat{s}, t, u)] \tag{7}$$

where $\mathrm{d}\hat{\sigma}^{(0+1)}$ comprises the NLO cross section including weak corrections and $\mathrm{d}\hat{\sigma}_{\mathrm{QED}}$ describes the QED part, i.e. virtual and real photon emission off the quarks and charged leptons. The PDFs depend on the QCD renormalization and factorization scales which we choose to be equal, denoted by Q^2. The radiation of collinear photons off quarks requires the factorization of the arising mass singularities into the PDFs which introduces a QED factorization scale μ_{QED} as will be explained in detail in the next section.

A. QED corrections

QED radiative corrections consist of the emission of real and virtual photons off the quarks and charged leptons. The $\mathcal{O}(\alpha)$ QED corrections to $q\bar{q} \to \gamma, Z \to l^+l^-$ can be further divided into gauge invariant subsets corresponding to initial and final-state radiation. The initial-state QED corrections contain quark-mass singularities, i.e. terms of the form $\ln(\hat{s}/m_q^2)$, which factorize and therefore can be absorbed by a redefinition (*renormalization*) of parton distribution functions (PDFs) [11]. This can be done in complete analogy to the calculation of QCD radiative corrections. By the redefinition, the mass singularities disappear from the observable cross sections and the renormalized distribution functions become dependent on the QED factorization scale μ_{QED} which is controlled by the well-known Gribov-Lipatov-Altarelli-Parisi (GLAP) equations [12]. These universal photonic corrections can be taken into account by a straightforward modification [13,14] of the standard GLAP equations which describe gluonic corrections only. The modification corresponds to the addition of a term of the order of the electromagnetic fine-structure constant α, resulting in modified distribution functions $q_f(x, \mu_{QED}^2)$ for quarks with flavour f. The gluon distribution $g(x, \mu_{QED}^2)$ is affected by QED corrections as well, although only indirectly, by terms of the order of $\mathcal{O}(\alpha\alpha_s)$. The QED factorization scale should be identified with a typical scale of the process, *i.e.* a large transverse momentum or the mass of a produced particle.

The proper treatment of the mass-singular initial-state QED corrections would require not only the solution of the evolution equations including the QED term, but also to correct all data that are used to fit the parton distributions for those QED effects. Apart from a few exceptions, experimental data have not been corrected for photon emission from quarks. However, one can illustrate the effect of the QED radiative corrections by comparing the modification of the parton distributions relative to the distribution functions obtained from the evolution equations without the QED terms, which are used as an input. For instance, in the context of the LHC workshop [5], taking the input distributions from [15] one found small, negative corrections at the per-mille level for all values of x and μ_{QED}^2 relevant in the LHC experiments. Only at large $x \gtrsim 0.5$ and large $\mu_{QED}^2 \gtrsim 10^3$ GeV^2 do the corrections reach the magnitude of one per cent. Other input distribution functions lead to differences of QED corrections at the per-mille level, which are again irrelevant when compared with the expected experimental precision of structure-function measurements.

The QED part of the complete $\mathcal{O}(\alpha)$ corrections implemented in ZGRAD2 is based on the complete calculation of the QED $\mathcal{O}(\alpha)$ radiative corrections to $pp(p\bar{p}) \to Z, \gamma \to l^+l^-(l = e, \mu)$ as carried out in [20]. The external fermions are considered to be massless and their masses are only kept to regularize the arising collinear singularities, i.e. they only appear in terms of the form $\ln(\hat{s}/m_f^2)$. The collinear singularities associated with initial-state photon radiation are factorized into the parton distribution functions as described above. Only when using MRST2004 QED the QED corrections are taken into account into the GLAP evolution of the PDFs.

B. Non-QED corrections and the effective Born approximation

The non-QED corrections comprise the remaining non-photonic virtual corrections, i.e. weak corrections such as: self-energy contributions to the photon and Z propagators, vertex corrections to the γ/Z-l^+l^- and γ/Z-$q\bar{q}$ couplings, and box diagrams with two massive gauge bosons. Since we consider all external fermions, quarks and leptons, to be massless, there is no Higgs boson contribution to the box diagrams and vertex corrections. The calculation of the radiative corrections is performed in the 't Hooft-Feynman gauge. To regularize and remove the arising UV divergences we use dimensional regularization in the ON-SHELL renormalization scheme as described in [16].

In the following we closely follow [19,17], especially for a careful treatment of higher-order corrections, which is important for a precise description of the Z resonance.

The NLO differential parton cross section of Eq. 7 including weak $\mathcal{O}(\alpha)$ and leading $\mathcal{O}(\alpha^2)$ corrections is of the following form

$$d\hat{\sigma}^{(0+1)} = d\mathrm{P}_{2\mathrm{f}} \, \frac{1}{12} \sum |A_\gamma^{(0+1)} + A_Z^{(0+1)}|^2(\hat{s},t,u) + d\hat{\sigma}_{\mathrm{box}}(\hat{s},t,u) \; . \tag{8}$$

$d\hat{\sigma}_{\mathrm{box}}$ describes the contribution of the box diagrams. The matrix elements $A_{\gamma,Z}^{(0+1)}$ comprise the Born matrix elements, $A_{\gamma,Z}^0$, the $\gamma, Z, \gamma Z$ self energy insertions, including a leading log resummation of the terms involving the light fermions, and the one-loop vertex corrections. $A_{\gamma,Z}^{(0+1)}$ can be expressed in terms of effective vector and axial-vector couplings $g_{V,A}^{(\gamma,Z),f}, f = l,q$, so that the matrix elements squared for massles external fermions read as follows:

$$\sum |A_\gamma^{(0+1)}|^2 = \frac{(4\pi\alpha)^2}{(1+\mathcal{R}e\hat{\Pi}^\gamma(\hat{s}))^2\,\hat{s}^2} \times$$
$$8\Big[(|g_V^{\gamma,l}|^2 + |g_A^{\gamma,l}|^2)\,(|g_V^{\gamma,q}|^2 + |g_A^{\gamma,q}|^2)\,(t^2+u^2)$$

$$
\begin{aligned}
&- 4\mathcal{R}e(g_V^{\gamma,l}(g_A^{\gamma,l})^*)\,\mathcal{R}e(g_V^{\gamma,q}(g_A^{\gamma,q})^*)\,(t^2-u^2)\Big] && (9)\\
\sum |A_Z^{(0+1)}|^2 = &\frac{|\chi(\hat{s})|^2}{(1+\mathcal{R}e\hat{\Pi}^Z(\hat{s}))^2\,\hat{s}^2}\times \\
&8\Big[(|g_V^{Z,l}|^2+|g_A^{Z,l}|^2)\,(|g_V^{Z,q}|^2+|g_A^{Z,q}|^2)\,(t^2+u^2) \\
&- 4\mathcal{R}e(g_V^{Z,l}(g_A^{Z,l})^*)\,\mathcal{R}e(g_V^{Z,q}(g_A^{Z,q})^*)\,(t^2-u^2)\Big] && (10)\\
\sum 2\mathcal{R}e(A_Z^{(0+1)}A_\gamma^{(0+1)*}) = &\frac{(4\pi\alpha)\,|\chi(\hat{s})|^2}{(1+\mathcal{R}e\hat{\Pi}^\gamma(\hat{s}))\,(1+\mathcal{R}e\hat{\Pi}^Z(\hat{s}))\,\hat{s}^2}\,16\mathcal{R}e\Big(\chi^{-1}(\hat{s})\times \\
&\Big[(g_V^{Z,l}(g_V^{\gamma,l})^*+g_A^{Z,l}(g_A^{\gamma,l})^*)(g_V^{Z,q}(g_V^{\gamma,q})^*+g_A^{Z,q}(g_A^{\gamma,q})^*)(t^2+u^2) \\
&- (g_A^{Z,l}(g_V^{\gamma,l})^*+g_V^{Z,l}(g_A^{\gamma,l})^*)(g_A^{Z,q}(g_V^{\gamma,q})^*+g_V^{Z,q}(g_A^{\gamma,q})^*)(t^2-u^2)\Big]\Big) && (11)
\end{aligned}
$$

with

$$
\begin{aligned}
g_A^{Z,f}(\hat{s}) &= a_f + G_A^{Z,f}(\hat{s}) \\
g_V^{Z,f}(\hat{s}) &= v_f + F_V^{Z,f}(\hat{s}) + Q_f\frac{\hat{\Pi}^{\gamma Z}(\hat{s})}{1+\hat{\Pi}^\gamma(\hat{s})} \\
g_A^{\gamma,f}(\hat{s}) &= -G_A^{\gamma,f}(\hat{s}) \\
g_V^{\gamma,f}(\hat{s}) &= Q_f - F_V^{\gamma,f}(\hat{s})\ . \qquad (12)
\end{aligned}
$$

$F_V^{(\gamma,Z),f}, G_A^{(\gamma,Z),f}$ denote the renormalized vector and axial-vector formfactors, which parametrize the weak corrections to the $(\gamma,Z)f\bar{f}$ vertices. $\hat{\Pi}^X, X=\gamma,Z,\gamma Z$ describe the renormalized photon, Z and (γ,Z) self energy insertions

$$
\hat{\Pi}^\gamma(\hat{s}) = \frac{\hat{\Sigma}^\gamma(\hat{s})}{\hat{s}} \qquad (13)
$$

$$
\hat{\Pi}^{\gamma Z}(\hat{s}) = \frac{\hat{\Sigma}^{\gamma Z}(\hat{s})}{\hat{s}} \qquad (14)
$$

$$
\hat{\Pi}^Z(\hat{s}) = \frac{1}{\hat{s}-M_Z^2}\Big(\hat{\Sigma}^Z(\hat{s}) - \frac{(\hat{\Sigma}^{\gamma Z}(\hat{s}))^2}{\hat{s}+\hat{\Sigma}^\gamma(\hat{s})}\Big)\ . \qquad (15)
$$

The box contribution $d\hat{\sigma}_{\rm box}$ cannot be absorbed in effective couplings. However, in the Z resonance region the box diagrams can be neglected and the NLO cross section $d\hat{\sigma}^{(0+1)}$ of Eq. 8 is of Born-structure. In Appendix B we describe the inclusion of leading higher-order (irreduzible) QCD and electroweak corrections connected to the ρ parameter.

Starting from the Born cross section of Eq. 1, an effective Born approximation (EBA) can be defined, which incorporates several entries from higher-order calculations as follows:

the effective (running) electromagnetic charge by replacing

$$\alpha \to \frac{\alpha}{1-\Delta\alpha(\hat{s})} , \quad \Delta\alpha(\hat{s}) = -\mathcal{R}e\hat{\Pi}^{\gamma}_{\rm ferm}(\hat{s}) , \tag{16}$$

where $\hat{\Pi}^{\gamma}_{ferm}$ denotes the fermion-loop contribution to the photon vacuum polarization; the Z propagator, together with the overall normalization factor of the neutral-current couplings in terms of the Fermi constant G_μ by using

$$\chi(\hat{s}) = 4\sqrt{2} G_\mu M_W^2 s_w^2 \, \frac{\hat{s}}{\hat{s} - M_Z^2 + {\rm i}\hat{s}\Gamma_Z/M_Z} , \tag{17}$$

containing the Z width as measured from the Z resonance at LEP; and the vertex and self energy corrections by replacing

$$v_f \to v_f^{\rm eff} = I_3^f - 2Q_f \sin^2\theta^f_{\rm eff} , \quad f = l, q , \tag{18}$$

containing the effective (leptonic) electroweak mixing angle at the Z peak, as measured at LEP and SLC. Taking Γ_Z and $\sin^2\theta^f_{\rm eff}$ from higher-order calculations

$$\sin^2\theta^f_{\rm eff} = \frac{1}{4|Q_f|} \Big(1 - \frac{\mathcal{R}e g_V^{Z,f}(M_Z^2)}{\mathcal{R}e g_A^{Z,f}(M_Z^2)}\Big) \tag{19}$$

with the effective couplings of Eq. 12 and $\Gamma_Z \equiv \Gamma_Z^{(0+1)}$ of Appendix A, for instance, yields a good description of the non-photonic electroweak corrections in the region around the Z resonance. A description in terms of an effective Born cross section far away from the Z pole becomes insufficient for two reasons: the effective couplings are not static but grow as functions of $\hat{s}$, and the box graphs are no longer negligible, but increase strongly with the energy and hence contribute sizeably at high invariant masses of the lepton pair.

C. Numerical discussion

For the numerical evaluation we chose the following set of SM input parameters: (needs to be updated !)

$$
\begin{aligned}
G_\mu = 1.16639 \times 10^{-5}\ \mathrm{GeV}^{-2}, &\qquad \alpha = 1/137.0359895, \quad \alpha_s \equiv \alpha_s(M_Z^2) = 0.119 \\
M_Z = 91.1867\ \mathrm{GeV}, &\qquad \Gamma_Z^{(0+1)} = 2.4932\ \mathrm{GeV} \\
m_e = 0.51099907\ \mathrm{keV}, &\qquad m_\mu = 0.105658389\ \mathrm{GeV}, \quad m_\tau = 1.777\ \mathrm{GeV} \\
m_u = 0.0464\ \mathrm{GeV}, &\qquad m_c = 1.5\ \mathrm{GeV}, \quad m_t = 174\ \mathrm{GeV} \\
m_d = 0.0465\ \mathrm{GeV}, &\qquad m_s = 0.15\ \mathrm{GeV}, \quad m_b = 4.7\ \mathrm{GeV}
\end{aligned} \tag{20}
$$

The W and Higgs boson masses, M_W and M_H, are related via loop corrections, which can be approximated as follows [21]

$$
\begin{aligned}
M_W = M_W^0 &- 0.0581\ \ln\Big(\frac{M_H}{100\mathrm{GeV}}\Big) - 0.0078 \ln^2\Big(\frac{M_H}{100\mathrm{GeV}}\Big) - 0.085\ \Big(\frac{\alpha_s}{0.118} - 1\Big) \\
&- 0.518\ \Big(\frac{\Delta\alpha_{had}^{(5)}(M_Z^2)}{0.028} - 1\Big) + 0.537\ \Big(\Big(\frac{m_t}{175\mathrm{GeV}}\Big)^2 - 1\Big)
\end{aligned} \tag{21}
$$

with $M_W^0 = 80.3805$ GeV. For the numerical discussion we choose $M_H = 120$ GeV. We work in the s-dependent width scheme and fix the weak mixing angle by $c_w = M_W/M_Z$, $s_w^2 = 1 - c_w^2$. The Z-boson decay width given above is calculated including electroweak and QCD corrections as described in Appendix A. The NLO prediction for the Z boson width is used throughout, i.e. also in the calculation of the lowest-order and EBA predictions. The fermion masses only enter through loop contributions to the vector boson self energies and as regulators of the collinear singularities which arise in the calculation of the QED contribution. The light quark masses are chosen in such a way, that the value for the hadronic five-flavour contribution to the photon vacuum polarization, $\Delta\alpha_{had}^{(5)}(M_Z^2) = 0.028$ [18], is recovered, which is derived from low-energy e^+e^- data with the help of dispersion relations. For this set of input parameters we obtain for the effective leptonic weak mixing angle of Eq. 19 $\sin^2\theta_{\mathrm{eff}}^l = 0.23167$.

APPENDIX A: THE Z DECAY WIDTH

The total Z decay width Γ_Z is obtained from the sum over the partial decay widths into fermion pairs as follows

$$\Gamma_Z = \sum_{f \neq t} \Gamma_{f\bar{f}} \,. \tag{A1}$$

At lowest order in perturbation theory the partial decay widths read

$$\Gamma^{(0)}_{f\bar{f}} = N^C_f \, \Gamma_0 \sqrt{1 - 4\mu_f} \left[(1 + 2\,\mu_f)\, v_f^2 + (1 - 4\,\mu_f)\, a_f^2 \right] \tag{A2}$$

with the color factor $N^C_f = 1, 3$, $f = l, q$ and

$$\Gamma_0 = \frac{\alpha\, M_Z}{3} \qquad \text{and} \qquad \mu_f = \frac{m_f^2}{M_Z^2} \,. \tag{A3}$$

The fermionic partial decay widths including electroweak and QCD radiative corrections can be expressed in terms of the effective coupling constants $g_f^{Z,V}, g_f^{Z,A}$ and the Z wave function renormalization contribution $\hat{\Pi}^Z$ of Eq. 12 and 15, respectively, as follows:

$$\Gamma^{(0+1)}_{f\bar{f}} = N^C_f \, \Gamma_0 \frac{\sqrt{1 - 4\mu_f}}{1 + \mathcal{R}e \hat{\Pi}^Z(M_Z^2)} \left[(1 + 2\,\mu_f)\, |g_V^{Z,f}(M_Z^2)|^2 + (1 - 4\,\mu_f)\, |g_A^{Z,f}(M_Z^2)|^2 \right] \\ \times (1 + \delta^f_{QED})\, (1 + \frac{N^f_C - 1}{2}\, \delta_{QCD}) \tag{A4}$$

The photonic QED corrections

$$\delta^f_{QED} = \frac{3\,\alpha\, Q_f^2}{4\,\pi} \tag{A5}$$

are very small, i.e. maximal 0.17% of the lowest-order decay width for charged leptons. The QCD corrections for massless hadronic final states have been calculated in [22,23] and can be parametrized as follows ($\alpha_s \equiv \alpha_s(M_Z^2)$)

$$\delta_{QCD} = \left(\frac{\alpha_s}{\pi} \right) + 1.405 \left(\frac{\alpha_s}{\pi} \right)^2 - 12.8 \left(\frac{\alpha_s}{\pi} \right)^3 - \frac{Q_f^2}{4} \frac{\alpha\, \alpha_s}{\pi^2} \,. \tag{A6}$$

The term $\mathcal{O}(\alpha\, \alpha_s)$ is also added although it is not a pure QCD contribution.

For b quarks the QCD corrections are different due to finite b mass terms and top-quark dependent 2-loop diagrams for the axial part. The calculation of the electroweak corrections to the Z decay widths assuming massless external fermions is also not a good approximations for the decay into τ lepton pairs. In order to take into account these effects we correct our results for the partial decay widths into b quarks and τ leptons as follows:

$$\Gamma_{b\bar{b}}^{(0+1)} = \Gamma_{b\bar{b}}^{(0+1)}(\mu_b = 0) - 0.0088 \text{ GeV}\,, \quad \Gamma_{\tau\tau}^{(0+1)} = \Gamma_{\tau\tau}^{(0+1)}(\mu_\tau = 0) - 0.00018 \text{ GeV}\,, \tag{A7}$$

which has been obtained by comparing with the complete calculation with massive external fermions of Ref. [25].

APPENDIX B: RENORMALIZED SELF ENERGIES AND FORM FACTORS

The renormalized self energies $\hat{\Sigma}^X(q^2), X = \gamma, Z, \gamma Z$ of the neutral vector bosons read

$$\hat{\Sigma}^\gamma(q^2) = \Sigma^\gamma(q^2) - q^2\,\Pi^\gamma(0) \tag{B1}$$

$$\begin{aligned}\hat{\Sigma}^Z(q^2) = \Sigma^Z(q^2) - \mathcal{R}e\Sigma^Z(M_Z^2) + (q^2 - M_Z^2)\Big[\frac{c_w^2 - s_w^2}{s_w^2}\Big(\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \\ - 2\frac{s_w}{c_w}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2}\Big) - \Pi^\gamma(0)\Big]\end{aligned} \tag{B2}$$

$$\hat{\Sigma}^{\gamma Z}(q^2) = \Sigma^{\gamma Z}(q^2) - \Sigma^{\gamma Z}(0) - q^2\frac{c_w}{s_w}\Big[\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} - 2\frac{s_w}{c_w}\frac{\Sigma^{\gamma Z}(0)}{M_Z^2}\Big] \tag{B3}$$

with $\Pi^\gamma(0) = \partial\Sigma^\gamma/\partial q^2(q^2 = 0)$ and the mass renormalization constants

$$\delta M_Z^2 = \mathcal{R}e\Big(\Sigma^Z(M_Z^2) - \frac{(\hat{\Sigma}^{\gamma Z}(M_Z^2))^2}{M_Z^2 + \hat{\Sigma}^\gamma(M_Z^2)}\Big), \quad \delta M_W^2 = \mathcal{R}e\Sigma^W(M_W^2) \tag{B4}$$

where δM_Z^2 is calculated via iteration. $\Sigma^X(q^2)$ $(X = \gamma, Z, \gamma Z, W)$ denote the unrenormalized self energies as the transverse coefficients in the expansion

$$\Sigma_{\mu\nu}^X(q^2) = -g_{\mu\nu}\Sigma^X(q^2) + \frac{q_\mu q_\nu}{q^2}\left[\Sigma^X(q^2) - \Sigma_L^X(q^2)\right]\,. \tag{B5}$$

The $q_\mu q_\nu$-terms yield only contributions $\propto m_f^2$ in the on-shell amplitudes and hence vanish in the limit $m_f \to 0$. Explicit expressions for the unrenormalized vector boson self energies $\Sigma^X, X = \gamma, Z, \gamma Z, W$ and for the renormalized form factors $F_V^{(Z,\gamma),f}, G_A^{(Z,\gamma),f}$ are provided in Appendix B and C.1 of Ref. [17], respectively.

Higher-order (irreducible) corrections connected to the ρ parameter are also taken into account by performing the replacement

$$\frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} \rightarrow \frac{\delta M_Z^2}{M_Z^2} - \frac{\delta M_W^2}{M_W^2} - \Delta\rho^{HO} \tag{B6}$$

in Eqs. B2,B3, where

$$\Delta\rho^{HO} = 3\frac{G_\mu m_t^2}{8\pi^2\sqrt{2}} \left[\frac{G_\mu m_t^2}{8\pi^2\sqrt{2}}\, \Delta\rho^{(2)}(m_t^2/M_H^2) + c_1\, \frac{\alpha_s(m_t)}{\pi} + c_2 \left(\frac{\alpha_s(m_t)}{\pi}\right)^2\right] . \tag{B7}$$

The coefficients c_1 and c_2 describe the first and second-order QCD corrections to the leading $G_\mu m_t^2$ contribution to the ρ parameter, calculated in [26] and [27], respectively. Their explicit expressions can be found in the Electroweak Working Group Report of [19] (Eqs. (83,84)). $\alpha_s(m_t^2)$ is calculated from $\alpha_s(M_Z^2)$ as follows (5 active flavors):

$$\alpha_s(m_t^2) = \frac{12\pi}{23}\, \frac{1}{\left[\ln\left(\frac{m_t^2}{M_Z^2}\right) + \frac{12\pi}{23\alpha_s(M_Z^2)}\right]} . \tag{B8}$$

The function $\Delta\rho^{(2)}(m_t^2/M_H^2)$ describes the leading two-loop electroweak corrections to the ρ parameter and is explicitly given in [28].

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